Practice Exam 1 Solutions

1.) let $f: X \to X$. S'pose $f \circ f = id|_X$ i.e. $(f \circ f)(x) = x$, for all $x \in X$. Show f is injective.

pf.

s'pose $f(x_1) = f(x_2)$ then applying f to both sides yields $(f \circ f)(x_1) = (f \circ f)(x_2)$ but this says $x_1 = x_2$. So f is 1-1. $\square$

2.) let $f: X \to Y$. Given fncs. $g, h: W \to X$ whenever $f \circ g = f \circ h$ then $g = h$. Show f is injective.

pf.

consider $g(w_1)$ and $h(w_2)$ for $w_1, w_2 \in W$. Since $g, h: W \to X$ then $g(w_1), h(w_2) \in X$ set $x_1 = g(w_1)$, $x_2 = h(w_2)$ consider $f(x_1) = f(x_2)$. This says $f(g(w_1)) = f(h(w_2)) \Rightarrow (f \circ g)(w_1) = (f \circ h)(w_2)$, but by hypothesis $\Rightarrow g = h$ So $g(w_1) = h(w_2) \Rightarrow x_1 = x_2$ so f is 1-1. $\square$

3.) let $f: X \to Y$ and $V_\alpha \subseteq Y$ for all $\alpha \in A$. Show $f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$

pf.

pick $x \in f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) \Rightarrow f(x) \in \bigcup_{\alpha \in A} V_\alpha \Rightarrow f(x) \in V_\alpha$ for some $\alpha \in A$

$\Rightarrow x \in f^{-1}(V_\alpha)$ for some $\alpha \in A \Rightarrow x \in \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$ so $f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) \subseteq \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$.

Now pick $x \in \bigcup_{\alpha \in A} f^{-1}(V_\alpha) \Rightarrow x \in f^{-1}(V_\alpha)$ for some $\alpha \in A \Rightarrow f(x) \in V_\alpha$ for some $\alpha \in A$

$\Rightarrow f(x) \in \bigcup_{\alpha \in A} V_\alpha$ ~~..~~. $\Rightarrow x \in f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right)$ so $\bigcup_{\alpha \in A} f^{-1}(V_\alpha) \subseteq f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right)$ showing set equality $\square$

4.) let $f: X \to Y$ and $V_\alpha \subseteq X$ for all $\alpha \in A$. Show $f\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f(V_\alpha)$

pf.

pick $f(x) \in f\left(\bigcup_{\alpha \in A} V_\alpha\right) \Rightarrow x \in \bigcup_{\alpha \in A} V_\alpha \Rightarrow x \in V_\alpha$ for some $\alpha \in A$. $\Rightarrow f(x) \in f(V_\alpha)$ for some $\alpha \in A$

$\Rightarrow f(x) \in \bigcup_{\alpha \in A} f(V_\alpha) \Rightarrow f\left(\bigcup_{\alpha \in A} V_\alpha\right) \subseteq \bigcup_{\alpha \in A} f(V_\alpha)$. Now pick $f(x) \in \bigcup_{\alpha \in A} f(V_\alpha)$

$\Rightarrow f(x) \in f(V_\alpha)$ for some $\alpha \in A$. $\Rightarrow x \in V_\alpha$ for some $\alpha \in A \Rightarrow x \in \bigcup_{\alpha \in A} V_\alpha$

$\Rightarrow f(x) \in f\left(\bigcup_{\alpha \in A} V_\alpha\right) \Rightarrow \bigcup_{\alpha \in A} f(V_\alpha) \subseteq f\left(\bigcup_{\alpha \in A} V_\alpha\right)$. showing set equality. $\square$

5.) let $\sim$ be on $X = \mathbb{Z} \times \mathbb{N}^+$ by $(a,b) \sim (c,d)$ iff $ad = bc$. show $\sim$ is equiv. relation

pf:

Need to check reflexive, symmetric and transitive. For reflexive consider $(a,b) \sim (a,b)$ the condition says $ab = ba$ since $a, b \in \mathbb{Z}$ & we know multiplication commutes so true. For symmetric consider $(a,b) \sim (c,d)$. so we get $ad = bc \Rightarrow bc = ad \Rightarrow cb = da \Rightarrow (c,d) \sim (a,b)$. So symmetry holds. Finally for transitive consider $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y)$. The condition says $ad = bc$ and $cy = dx$. then $ad = bc \Rightarrow ady = bcy$ so $ady = bdx \Rightarrow ay = bx$ so $(a,b) \sim (x,y)$ implying transitive. $\square$

6.) let $\sim$ be on $\mathbb{R}$ by $x \sim y$ iff $|x| = |y|$. show $\sim$ is equiv. relation.

pf:

Need to check reflexive, symmetry and transitive. For reflexive consider $x \sim x$ so condition says $|x| = |x|$, this always holds. so reflexive is true. Next for symmetry consider $x \sim y \Rightarrow |x| = |y| \Rightarrow |y| = |x|$ so get $y \sim x$. so symmetric true. Finally for transitive, let $x \sim y$ and $y \sim z$ so condition says $|x| = |y|$ and $|y| = |z|$ so $|x| = |y| = |z| \Rightarrow |x| = |z|$. $\Rightarrow x \sim z$. so transitive $\square$

7.) what are the multiplication / addition tables for $\mathbb{Z}/7\mathbb{Z}$.

pf:

1st note $\mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\}$ and operations in $\mathbb{Z}/7\mathbb{Z}$ are done w/ congruence mod 7 meaning if $x \geq 7$ we find the remainder upon division. so $4 \cdot 5 = 20 \equiv_7 6$ etc. so

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

8.) What are the multiplication/addition tables for $\mathbb{Z}/4\mathbb{Z}$

pf:

1st note $\mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$ and operations in $\mathbb{Z}/4\mathbb{Z}$ are done w/ congruence mod 4 means if $x \geq 4$ then we find the remainder upon division of 4 so $3 \cdot 7 = 21 \equiv_4 1$ etc.

+	0	1	2	3
0	0	1	2	3
1	[illegible]	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

9.) let $f: X \to Y$ and $V_\alpha \subseteq X$ for all $\alpha \in A$. Show $f(\bigcap_{\alpha \in A} V_\alpha) \subseteq \bigcap_{\alpha \in A} f(V_\alpha)$.

pf:

pick $f(x) \in f(\bigcap_{\alpha \in A} V_\alpha) \Rightarrow x \in \bigcap_{\alpha \in A} V_\alpha \Rightarrow x \in V_\alpha$ for all $\alpha \in A$. $\Rightarrow f(x) \in f(V_\alpha)$ for all $\alpha \in A$

$\Rightarrow f(x) \in \bigcap_{\alpha \in A} f(V_\alpha)$. so $f(\bigcap_{\alpha \in A} V_\alpha) \subseteq \bigcap_{\alpha \in A} f(V_\alpha)$. $\square$